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Nonlinear Longitudinal Stability of a Symmetric Aircraft

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The longitudinal stability of a symmetric aircraft is examined beyond the well-known linearization about a steady mean state, leading to the phugoid and short-period modes. The exact equations of longitudinal motion of a symmetric aircraft are considered, i.e., balance of longitudinal and transverse force (without side force) and balance of pitching moment; the drag terms included are friction and lift-induced drag, plus nonsymmetric lift–drag polar; the mass density is taken as a constant, as well as thrust along the flight path. Elimination would lead to a fourth-order nonlinear differential equation for the angle of attack relative to the angle of zero pitching moment, if the acceleration of the flight-path angle is neglected, and the flight-path angle is moderate. In the case of a small flight-path angle, it simplifies to a third-order differential equation, containing a set of nonlinear corrections, to a second-order linear equation, specifying sinusoidal oscillation of the relative angle of attack. It is shown, by a small perturbation method, that forced oscillations occur at its harmonics (viz. double or triple frequency), and free oscillations can have decaying or growing amplitude. In the case of statically stable aircraft, the oscillations have a short period and grow or decay slowly. In the case of statically unstable aircraft the growth is rapid, as in a pilot-induced oscillation.

Nomenclature

A = amplitude of free zero-order oscillation, Eq. (42b)
a = constant parameter, Eq. (10b)
B = amplitude of free first-order oscillations, Eq. (54a)
b = thrust-to-mass ratio, Eq. (15a)
$\bar{b}$ = wing span, m
C_D = drag coefficient
C_{Df} = friction drag coefficient
$C_{D\alpha}$ = slope of drag coefficient
C_{D0} = drag coefficient at zero angle of attack
C_L = lift coefficient
$C_{L\alpha}$ = slope of lift coefficient, rad^{-1}
C_{L0} = lift coefficient at zero angle of attack
C_M = pitching moment coefficient
$C_{M\alpha}$ = slope of pitching moment coefficient
C_{M0} = pitching moment coefficient at zero angle of attack
C_1, C_2, C_3 = amplitudes of forced first-order oscillations, Eq. (49)
$C_\pm$, C_* = amplitudes of free first-order oscillations, Eq. (55)
c = mean aerodynamical chord, m
D = drag force, kN
F = nonlinear return force, Eq. (44)
f = constant parameter, Eq. (26a), m^{-1}
$\bar{f}$ = constant parameter, Eq. (15b), m^{-1}
f_0 = parameter in thrust as function of airspeed, Eq. (16a)
g = acceleration of gravity, 9.81 m s^{-2}
h = constant parameter, Eq. (26b)
$\bar{h}$ = constant parameter, Eq. (15c)
I = transverse moment of inertia, kg m^2
k = coefficient of lift-induced drag, Eq. (5)
L = lift force
M = pitching moment
m = mass, kg
N = number of cycles to double or halve amplitude
q = kinetic energy per unit mass, Eq. (20a)
r = constant parameter, Eq. (41)
S = reference or wing area, m^2
s = distance along flight path
T = thrust, kN
t = time
U = airspeed, m/s
W = weight
α = angle of attack, rad
α_0 = angle of attack for zero lift, rad, Eq. (4b)
α_1 = angle of attack for zero pitching moment, rad, Eq. (6b)
γ = flight-path angle, rad
ε = constant parameter, Eq. (26c)
$\bar{\varepsilon}$ = constant parameter, Eq. (15d)
θ = angle of attack relative to angle of attack for zero lift, rad, Eq. (11a)
θ_1 = angle of attack for zero pitching moment relative to angle of attack for zero lift, rad, Eq. (18b)
λ = coefficient in nonparabolic lift–drag polar, Eq. (5)
μ = constant nonlinearity parameter, Eq. (45)
ρ = mass density, kg m^{-3}
τ = time to double or halve amplitude, s
τ_0 = period of oscillation, s
ϑ = growth or decay rate of oscillations, s^{-1}, Eq. (56)
ς = angle of attack relative to angle of attack for zero pitching moment, rad, Eq. (23b)
$\dot{\varsigma}$ = first-order time derivative of ς
$\ddot{\varsigma}$ = second-order time derivative of ς
$\dddot{\varsigma}$ = third-order time derivative of ς
$\ddddot{\varsigma}$ = fourth-order time derivative of ς
ς_0 = relative angle of attack for fundamental oscillation, rad, Eq. (42a)
ς_1 = relative angle of attack for first-order perturbation, rad, Eq. (46a)
$\bar{\varsigma}_1$ = relative angle of attack for free, first-order oscillation, Eq. (54a)
ω = frequency of oscillation, rad s^{-1}, Eq. (46b)

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ω_0	= frequency of fundamental oscillation, rad s^{-1}, Eq. (39)
ω_1	= perturbation of frequency of oscillation, Eq. (46b)
ω_*	= frequency of angle-of-attack oscillations, Eq. (38b)

Introduction

THE equations of the longitudinal motion of a symmetric aeroplane are usually solved in the literature,[1-10] using the method[11] of linearization of the fourth-order system about a mean state of steady flight, to obtain the frequency and damping of the phugoid and short-period modes. An even older approach[12] is to solve exactly the nonlinear equations of motion, as shown by the phugoid.[13] The full system of nonlinear equations of longitudinal stability is quite complex, and its solution has been approached by the method of bifurcations, together with numerical methods, viz., in the investigation of spins.[14-20] Explicit analytic solutions are rare; for example, one is the inverse phugoid problem, concerning the nonlinear perturbation of flight along a constant glide slope.[21-24] There is flight test data that support the latter theory of nonlinear stability with glide slope constraint,[25] as there are data for parameter identification methods.[26-29] This case involves only one degree of freedom, and nonlinear stability with three degrees of freedom can be approached by an iterative, cyclic method,[30] leading to a sequence of linear problems with variable coefficients, which can be solved in terms of special functions, Bessel confluent, and Gaussian hypergeometric types. Linear equations with coefficients dependent on time occur for flights in the presence of atmospheric perturbations, like gusts and windshears.[31-52] In the present paper the problem of nonlinear stability with three degrees of freedom is addressed directly, by elimination between the fourth-order system of equations of motion, to obtain a single higher-order equation for angle of attack. The approach is thus quite different from the phugoid, in the assumptions made:

1) The phugoid assumes flight at constant angle of attack, with drag equal to thrust, and at unrestricted flight path angle; for example, the loop is included.

2) The present analysis restricts the flight-path angle and concentrates on the dynamics of the angle of attack, without requiring drag to balance thrust.

Thus, the present nonlinear analysis of longitudinal aircraft stability is, in a sense, complementary to the theory of the phugoid.

The exact balance of lift, drag, weight, thrust, and inertia force is taken, together with the pitching moment equation, under the assumptions of flight at low Mach number, away from the stall, for constant air density and constant thrust along the flight path. This leads to a fourth-order system of coupled nonlinear differential equations, which can be written using as an independent variable either the time or distance along the flight path. The dependent variables are the flight-path angle, angle of attack, and airspeed; if the latter is replaced by airspeed squared or kinetic energy per unit mass the force balance equations become more compact, unlike the pitching moment equation. Elimination between these equations, with neglect of the acceleration of flight-path angle, leads (case I) to a rather complicated nonlinear, fourth-order differential equation for angle of attack. By restricting γ to moderate values (case II), $\gamma \leq 30$ deg (so that $\gamma^2 << 1$, rad), the flight-path angle equation decouples, but the fourth-order nonlinear differential equation for angle of attack still remains fairly complicated. It is simplified further for small flight-path angle $\gamma \leq 4$–10 deg (or $\gamma << 1$, rad) (case III), when it becomes a third-order differential equation for angle of attack, consisting of 1) linear terms, forming a second-order differential equation with constant fundamental frequency, independent of airspeed, as for the short-period mode; 2) one set of nonlinear terms, associated with nonparabolic lift drag polar and lift-induced drag, which introduce, respectively, quadratic and cubic powers of the angle of attack; and 3) the other set of nonlinear terms consists of products of powers of derivatives of angle of attack, which arise from the elimination among the equations of motion, and thus could be loosely interpreted as a coupling to the phugoid motion.

The method of solution that is used for case III, and would apply as well (with more tedious algebra) to cases I and II, is to consider a perturbation of the simple fundamental mode, with constant amplitude A and frequency ω_0. It is found that the nonlinear effects do not cause a change in oscillation frequency and induce oscillations at multiples of the fundamental frequency. The calculation, to the lowest order in the perturbation parameter of the amplitude and phase of the oscillation induced at the harmonics of the short-period frequency, can be made by perturbing the fundamental mode and linearizing the third-order differential equation for angle of attack. The forced solution then specifies an excitation at the double and triple of the fundamental frequency, whereas the free oscillations lead to a cubic equation for the frequency ω. The roots of this cubic equation are, besides the fundamental period, a complex frequency whose imaginary part specifies an amplified or damped mode. The fundamental frequency is short and the number of cycles to double amplitude long, for statically stable aircraft, whether of fighter or transport type. In the case of statically unstable aircraft, the roles of frequency and damping are interchanged, and thus the time scales for instability are short, leading to a rapid growth of angle-of-attack oscillations, as in pilot-induced oscillations (PIOs).

Fourth-Order System of Equations for Longitudinal Stability of Symmetric Aircraft

The equations of longitudinal motion of a symmetric aircraft are written, using as independent variable time or distance along the flight path, and as dependent variables angle of attack, flight-path angle and airspeed, or kinetic energy per unit mass.

Balance of Forces (Without Side Force) and Pitching Moment

Considering the longitudinal motion of a symmetric aircraft (Fig. 1), under L, D, T (along the flight path), and W, balancing the inertia force, and the force balance (Ref. 12, page 540) along the tangent [Eq. (1a)] and normal [Eq. (1b)] to the flight paths can be written

$$m\dot{U} = T - D - W \sin\gamma \tag{1a}$$

$$mU\dot{\gamma} = L - W\cos\gamma \tag{1b}$$

where the acceleration is written in tangential $\dot{U}$ and centripetal $U\dot{\gamma}$ components and the overdot denotes time derivative; for example, it appears twice applied to α in the M equation:

$$\dot{f} \equiv \frac{\partial f}{\partial t}: \quad I(\ddot{\alpha} + \ddot{\gamma}) = M \tag{2}$$

where I denotes the moment of inertia relative to an axis transverse to the plane of motion (or oscullating plane of the trajectory, defined by the tangent and normal to the trajectory).

The system of equations of motion [Eqs. (1a), (1b), and (2)] is of the fourth-order, and its couplings and nonlinearities are specified not only by the dynamics of rigid bodies, but also by the aerodynamic laws, for L, D, and M:

$$L = \tfrac{1}{2}\rho S U^2 C_L(\alpha) \tag{3a}$$

$$D = \tfrac{1}{2}\rho S U^2 C_D(\alpha) \tag{3b}$$

$$M = \tfrac{1}{2}\rho c S U^2 C_M(\alpha) \tag{3c}$$

where ρ, S (the reference area), and c all are taken as constant (no change in aircraft configuration and small altitude excursions), and for flight at low Mach number, the lift C_L, drag C_D, and pitching moment C_M coefficients depend only on angle of

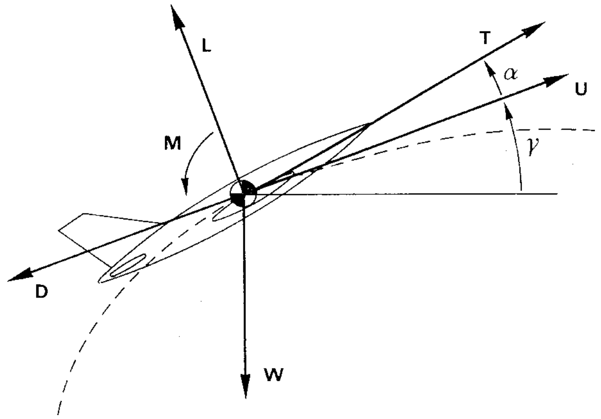

Fig. 1 Balance of forces on the oscullating plane of the flight path, and of pitching moment orthogonal to this plane.

attack, viz., for flight away from the stall: 1) the lift coefficient is a linear function of angle of attack:

$$C_L(\alpha) = C_{L0} + \alpha C_{L\alpha} = C_{L\alpha}(\alpha - \alpha_0) \tag{4a}$$

with value C_{L0} at zero angle of attack and slope $C_{L\alpha}$, thus vanishing at α_0

$$\alpha_0 \equiv -C_{L0}/C_{L\alpha} \tag{4b}$$

2) the drag coefficient is because of C_{Df} and k, and we also add a nonparabolic lift–drag polar term, with coefficient λ

$$C_D(\alpha) = C_{D0} + \lambda C_L(\alpha) + k[C_L(\alpha)]^2 \tag{5}$$

3) the pitching moment coefficient is again a linear function of angle of attack

$$C_M(\alpha) = C_{M0} + \alpha C_{M\alpha} = C_{M\alpha}(\alpha - \alpha_1) \tag{6a}$$

with value C_{M0} at zero angle of attack, and slope $C_{M\alpha}$, thus vanishing at the angle

$$\alpha_1 \equiv -C_{M0}/C_{M\alpha} \tag{6b}$$

of zero pitching moment.

Time or Distance Along Flight Path as Independent Variable

Denoting by prime derivative with regard to s

$$f' \equiv \frac{\partial f}{\partial s}: \quad \dot{f} \equiv \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial s}\right)\frac{\mathrm{d}s}{\mathrm{d}t} = Uf' \tag{7a}$$

and using Eq. (7a) together with

$$\ddot{f} = \left(U\frac{\mathrm{d}}{\mathrm{d}s}\right)^2 f = \left(U\frac{\mathrm{d}}{\mathrm{d}s}\right) Uf' = U^2 f'' + U'Uf' \tag{7b}$$

the equations of motion [Eqs. (1a), (1b), and (2)] are rewritten

$$mUU' = T - D - W\sin\gamma \tag{8a}$$

$$mU^2\gamma' = L - W\cos\gamma \tag{8b}$$

$$U^2(\alpha'' + \gamma'') + UU'(\alpha' + \gamma') = M/I \tag{8c}$$

in Eqs. (8a–8c) the independent variable is the distance along the flight path, instead of time in Eqs. (1a), (1b), and (2). To more easily perform eliminations among the fourth-order system of coupled nonlinear differential equations [Eqs. (8a–8c)], it is necessary to distinguish clearly the variables from the constant coefficients, arising from Eqs. (3a), (3b), (4a), (4b), (5), (6a), and (6b).

Starting with the radial force balance [Eq. (8b)]

$$\gamma' + (g/U^2)\cos\gamma = L/mU^2 = (\rho S/2m)C_{L\alpha}(\alpha - \alpha_0) \tag{9}$$

where Eqs. (3a) and (4a) were used, leading to

$$\gamma' = -gU^{-2}\cos\gamma + a\theta \tag{10a}$$

where a is a constant

$$a \equiv \rho S C_{L\alpha}/2m \tag{10b}$$

and θ

$$\theta \equiv \alpha - \alpha_0 \tag{11a}$$

in terms of which the lift coefficient can be written

$$C_L(\theta) = C_{L\alpha}\theta \tag{11b}$$

Similarly, in the tangential force balance [Eq. (8a)]

$$U'U + g\sin\gamma - T/m = -D/m = -(\rho S/2m)U^2 C_D \tag{12}$$

Eqs. (3b), (5), and (11b) can be used

$$U'U + g\sin\gamma - T/m = -(\rho S/2m)U^2(C_{D0} + \lambda C_{L\alpha}\theta + kC_{L\alpha}^2\theta^2) \tag{13}$$

to obtain

$$U'U = b - g\sin\gamma - \bar{f}U^2(1 + \bar{h}\theta + \bar{\varepsilon}\theta^2) \tag{14}$$

where

$$b \equiv T/m \tag{15a}$$

$$\bar{f} \equiv \rho S C_{D0}/2m \tag{15b}$$

$$\bar{h} \equiv \lambda C_{L\alpha}/C_{D0} \tag{15c}$$

$$\bar{\varepsilon} \equiv kC_{L\alpha}^2/C_{D0} \tag{15d}$$

the last two [Eqs. (15c) and (15d)] are dimensionless constants; Eq. (15b) is the ballistic coefficient associated with friction drag; and Eq. (15a) is the thrust per unit mass. The latter is assumed to be also constant, implying either no throttle movement or slow response compared with other time scales of the problem. The analysis would remain valid for a thrust per unit mass dependence on airspeed

$$T(U)/m = b - f_0 U^2 \tag{16a}$$

where the constant f_0 adds to Eq. (15b)

$$\bar{f} = f_0 + \rho S C_{D0}/2m \tag{16b}$$

and Eq. (15a) is unchanged, with b defined by Eq. (16a).

Airspeed or Kinetic Energy per Unit Mass as Dependent Variable

In the remaining equation of motion, for pitching moment [Eq. (8c)]

$$U^2(\alpha'' + \gamma'') + U'U(\alpha' + \gamma') = (\rho Sc/2I)U^2 C_M(\alpha) \tag{17a}$$

can be used [Eqs. (3c) and (6a)]

$$C_M(\alpha) = C_{M\alpha}(\alpha - \alpha_1) = C_{M\alpha}[(\alpha - \alpha_0) - (\alpha_1 - \alpha_0)] \quad (17b)$$

and also the angle of attack relative to the angle of attack for zero lift [Eq. (11a)]

$$C_M(\theta) = C_{M\alpha}(\theta - \theta_1) \quad (18a)$$

which is distinct from the angle of attack for zero pitching moment

$$\theta_1 \equiv \alpha_1 - \alpha_0 \quad (18b)$$

substitution of Eqs. (11a) and (18a) in Eq. (17a) yields

$$O = U^2(\theta'' + \gamma'') + U'U(\theta' + \gamma') + jU^2(\theta - \theta_1) \quad (19a)$$

where j is again a constant

$$j \equiv -\rho c S C_{M\alpha}/2I \quad (19b)$$

In the fourth-order nonlinear coupled system of differential equations of motion [Eqs. (10a), (14), and (19a)], U, γ, and θ appear as functions of distance along the flight path.

U could be replaced, as a dependent variable, by the kinetic energy per unit mass

$$q \equiv \tfrac{1}{2}U^2 \quad (20a)$$

$$2U'/U = q'/q \quad (20b)$$

thereby simplifying somewhat the two balance-of-forces equations [Eqs. (21a) and (21b) ≡ (14) and (10a)] in the system

$$q'/2 = b - g\sin\gamma - \bar{f}q(1 + \bar{h}\theta + \bar{\varepsilon}\theta^2) \quad (21a)$$

$$\gamma' = -(g/q)\cos\gamma + a\theta \quad (21b)$$

$$(\theta'' + \gamma'') + (q'/2q)(\theta' + \gamma') + j(\theta - \theta_1) = 0 \quad (21c)$$

The pitching moment Eq. (21c) is simplest using time as the independent variable:

$$\dot{U} = b - g\sin\gamma - \bar{f}U^2(1 + \bar{h}\theta + \bar{\varepsilon}\theta^2) \quad (22a)$$

$$\dot{\gamma} = -gU^{-1}\cos\gamma + aU\theta \quad (22b)$$

$$\ddot{\theta} + \ddot{\gamma} + jU^2(\theta - \theta_1) = 0 \quad (22c)$$

and returning to the airspeed as dependent variable, the system (21a–21c) becomes (22a–22c).

Decoupling of Flight-Path Angle and Elimination for a Third-Order Equation

Neglect the acceleration $\ddot{\gamma}$ of γ in Eq. (22c), when eliminating among the equations of motion leads to nonlinear differential equation for angle of attack, which is of order four, for general flight-path angle; this remains of order four for moderate $\gamma \le 30$-deg flight-path angle and reduces to order three for small $\gamma \le 4$–10-deg flight-path angle. Even in the latter, simplest case, the nonlinear terms are quite evident.

Elimination for Small or Moderate Flight-Path Angle

Subsequently, three approximations will be considered, as concerns the flight-path angle, viz., 1) $\ddot{\gamma} \ll \ddot{\alpha}$, 2) $\gamma^2 \ll 1$, and 3) $\gamma \ll T/W$, whose implications are discussed next. The assumption 1 of acceleration of flight path angle negligible relative to acceleration of angle-of-attack $\ddot{\gamma} \ll \ddot{\alpha}$ restricts the kind of flight maneuvers allowed; for example, it holds for 1) a descent on a nearly constant glide slope, in which case some pitch activity may be required to keep the flight slope and 2) horizontal flight, for example, accelerating or decelerating. It would not hold for other maneuvers, like rapid climbs or descents. The assumption 2 of moderate flight-path angle $\gamma^2 \ll 1$, i.e., not exceeding $\gamma < 30$ deg, excludes steep climbs or descents, for example, a loop would be excluded. A more restrictive assumption is 3, small flight-path angle relative to the thrust-to-weight ratio $\gamma \ll T/W$. This assumption is not met by existing aircraft in all conditions, and thus, is again a restriction on maneuvering, as will be shown by considering two examples. For a modern high-agility fighter, the thrust-to-weight ratio in air combat configuration may reach $T/W \sim 1$ to 1.3; these air superiority fighters can climb vertically $\gamma = 90$ deg, so that $\gamma = \pi/2 = 1.57$ rad and $\gamma \ll T/W$ is not met generally, but only for flight path angles corresponding to $\gamma \le 0.15$ or $\gamma \le 10$ deg. For a large jet transport $T/W \sim 0.2$ to 0.5, and the assumption $\gamma \ll T/W$ is even more restrictive, viz., $\gamma \le 0.04$ or $\gamma \le 4$ deg. Thus, the range of flight-path angles is restricted, though less so for the high-agility fighter case. In spite of this, it is still worthwhile to study nonlinear effects, since the methods used extend from case 3 to 2 or 1, with greater analytical complexity.

To facilitate the elimination of the system of equations of motion, the pitching moment equation [Eq. (22c)] can be put in the simplest form

$$\ddot{\gamma} \ll \ddot{\varsigma}: \quad \ddot{\varsigma} + jU^2\varsigma = 0 \quad (23a)$$

by introducing the angle of attack relative to the angle of zero pitching moment:

$$\varsigma \equiv \theta - \theta_1 = \alpha - \alpha_1 \quad (23b)$$

besides neglecting the acceleration of the flight-path angle $\ddot{\gamma} \ll \ddot{\varsigma}$ relative to the acceleration of the angle of attack. In terms of the angle of attack relative to the angle of attack for zero pitching moment, which will now be called relative angle of attack, the force balance equations [Eqs. (22b) and (22c)] become

$$\dot{\gamma} + gU^{-1}\cos\gamma = aU(\varsigma + \theta_1) \quad (24a)$$

$$\dot{U} = b - g\sin\gamma - fU^2(1 + h\varsigma + \varepsilon\varsigma^2) \quad (24b)$$

where

$$f(1 + h\varsigma + \varepsilon\varsigma^2) = \bar{f}(1 + \bar{h}\theta + \bar{\varepsilon}\theta^2) = \bar{f}[1 + \bar{h}(\varsigma + \theta_1) + \bar{\varepsilon}(\varsigma + \theta_1)^2] \quad (25)$$

so that the new constants f, h, and ε are specified in terms of the old [Eqs. (15b–15d)] by

$$f \equiv \bar{f}(1 + \theta_1\bar{h} + \theta_1^2\bar{\varepsilon}) \quad (26a)$$

$$fh \equiv \bar{f}(\bar{h} + 2\theta_1\bar{\varepsilon}) \quad (26b)$$

$$f\varepsilon \equiv \bar{f}\bar{\varepsilon} \quad (26c)$$

By substitution of Eqs. (15b–15d) in Eq. (26a), it follows that f is given by

$$f = (\rho S/2m)(C_{D0} + \lambda C_{L\alpha}\theta_1 + kC_{L\alpha}^2\theta_1^2) = (\rho S/2m)C_D(\theta_1) \quad (26d)$$

where Eq. (5) was used.

The elimination among the equations of motion [Eqs. (23a), (24a), and (24b)] for the relative angle of attack without any further restriction (case 1) would lead to a nonlinear fourth-order differential equation, which is rather complicated. The deduction will be made in the simpler case 2 of moderate flight-path angle $\gamma \le 30$ deg, when its square (in radians) can be neglected:

$$\cos\gamma = 1 + \mathcal{O}(\gamma^2) \tag{27a}$$

$$\sin\gamma = \gamma + \mathcal{O}(\gamma^3) \tag{27b}$$

and γ no longer appears through trigonometric functions, but only algebraically, in the equations of motion [Eqs. (23a), (24a), and (24b)]:

$$\ddot{\varsigma} = -jU^2\varsigma \tag{28a}$$

$$\dot{\gamma} = aU(\varsigma + \theta_1) - g/U \tag{28b}$$

$$\dot{U} = b - g\gamma - fU^2(1 + h\varsigma + \varepsilon\varsigma^2) \tag{28c}$$

These will lead (case 2) to a simpler nonlinear differential equation for angle of attack (than for case 1), but still of fourth-order.

If the flight path is small $\gamma \le 10$ deg, in the precise sense (case 3) of negligible compared to the thrust-to-weight ratio,

$$\gamma << b/g = T/mg = T/W: \quad \dot{\gamma} = aU(\varsigma + \theta_1) - g/U \tag{29}$$

the flight-path angle is specified by the radial force balance equation [Eq. (29)], which decouples from the other equations, viz., the tangential force balance and pitching moment equations:

$$\ddot{\varsigma} = -jU^2\varsigma \tag{30a}$$

$$\dot{U} = b - fU^2(1 + h\varsigma + \varepsilon\varsigma^2) \tag{30b}$$

which lead to a nonlinear third-order differential equation for the angle of attack, discussed in the following text.

Fourth- and Third-Order Nonlinear Differential Equations for the Relative Angle of Attack

To perform the elimination of the system [Eqs. (28a–28c)] for a moderate flight-path angle, Eq. (28c) is differentiated (with regard to time)

$$\ddot{U} + 2fU\dot{U}(1 + \varepsilon\varsigma + h\varsigma^2) + fU^2(h + 2\varepsilon\varsigma)\dot{\varsigma} = -g\dot{\gamma} = g^2/U - agU(\varsigma + \theta_1) \tag{31}$$

and Eq. (28b) substituted, leading to a relation between U and ς, i.e., γ is absent; U can be specified from Eqs. (28a) $\equiv$ (32a):

$$U\sqrt{j} = \pm i\ddot{\varsigma}^{1/2}\varsigma^{-1/2} \tag{32a}$$

$$\mp 2i\dot{U}\sqrt{j} = \ddot{\varsigma}^{-1/2}\varsigma^{-1/2}\dddot{\varsigma} - \varsigma^{-3/2}\dot{\varsigma}\ddot{\varsigma}^{1/2} \tag{32b}$$

$$\mp 4i\ddot{U}\sqrt{j} = 2\ddot{\varsigma}^{-1/2}\varsigma^{-1/2}\ddddot{\varsigma} - 2\varsigma^{-3/2}\dot{\varsigma}\ddot{\varsigma}^{-1/2}\dddot{\varsigma} - \varsigma^{-1/2}\ddot{\varsigma}^{-3/2}\dddot{\varsigma}^2 - \varsigma^{-3/2}\ddot{\varsigma}^{3/2} + 3\varsigma^{-5/2}\dot{\varsigma}^2\ddot{\varsigma}^{1/2} \tag{32c}$$

together with its first two time derivatives [Eqs. (32b) and (32c)]. Note that in all terms of Eqs. (32a–32c), the powers of ς cancel; for example, in

$$\varsigma^{-3/2}\dot{\varsigma}\ddot{\varsigma}^{-1/2}\dddot{\varsigma}, \qquad \varsigma^{-5/2}\dot{\varsigma}^2\ddot{\varsigma}^{1/2} \tag{33}$$

the powers of ς add, respectively, $-\frac{3}{2} + 1 - \frac{1}{2} + 1 = 0 = -\frac{5}{2} + 2 + \frac{1}{2}$; also, orders of time derivatives add to $+1$ in Eq. (32a), to $+2$ in Eq. (32b) and to $+3$ in Eq. (32c); for example, in Eq. (33) the orders of time derivatives add, respectively, to $1 + 2 \times (-\frac{1}{2}) + 3 = 3 = 2 \times 1 + 2 \times (\frac{1}{2})$.

Substitution of Eqs. (32a–32c) into Eq. (31) leads to the following fourth-order nonlinear differential equation for the relative angle of attack

$$2\ddddot{\varsigma} + (\ddot{\varsigma}^{-1}\dddot{\varsigma} + 2\varsigma^{-1}\dot{\varsigma})\ddot{\varsigma}^{-1}\dddot{\varsigma} - \varsigma^{-1}\ddot{\varsigma} + 3\varsigma^{-2}\dot{\varsigma}^2 \pm (4f\sqrt{j})(1 + h\varsigma + \varepsilon\varsigma^2)(\ddot{\varsigma}^{-1/2}\varsigma^{-1/2}\dddot{\varsigma} - \varsigma^{-3/2}\dot{\varsigma}\ddot{\varsigma}^{1/2}) + 4g[a(\varsigma + \theta_1) + gj\varsigma\ddot{\varsigma}^{-1}] = 0 \tag{34}$$

After this is solved for $\varsigma(t)$, the $U(t)$ is given by Eq. (32a), and $\gamma(t)$ by integration of Eq. (28b). In the case of small flight-path angle [Eqs. (30a) and (30b)], it is sufficient to substitute Eqs. (32a) and (32b) in Eq. (30b), leading to a third-order nonlinear differential equation for the relative angle of attack

$$0 = \dddot{\varsigma} - \varsigma^{-1}\dot{\varsigma}\ddot{\varsigma} \pm 2ib\sqrt{j}\varsigma^{1/2}\ddot{\varsigma}^{1/2} \pm 2i(f/\sqrt{j})(1 + h\varsigma + \varepsilon\varsigma^2)\varsigma^{-1/2}\ddot{\varsigma}^{3/2} \tag{35}$$

once this is solved for $\varsigma(t)$, both $U(t)$ and $\gamma(t)$ are determined, as before, from Eqs. (32a) and (28b), respectively.

Nonsinusoidal Oscillation with Anharmonic and Other Nonlinear Terms

To interpret the differential equation for the relative angle of attack, Eq. (35) can be rewritten in the form

$$\ddot{\varsigma}[1 + h\varsigma + \varepsilon\varsigma^2] + (bj/f)\varsigma = \pm(i\sqrt{j}/2f)(\varsigma^{1/2}\ddot{\varsigma}^{-1/2}\dddot{\varsigma} - \varsigma^{-1/2}\dot{\varsigma}\ddot{\varsigma}^{1/2}) \tag{36}$$

From Eqs. (2), (3c), and (6a) it follows that

$$\ddot{\gamma} << \ddot{\alpha}: \quad \ddot{\alpha} - (\rho c S C_{M\alpha} U^2/2I)(\alpha - \alpha_1) = 0 \tag{37}$$

so that Eq. (23b)

$$\ddot{\varsigma} + \omega_*^2\varsigma = 0 \tag{38a}$$

where ω_* is for constant U:

$$\omega_*^2 \equiv -\rho c S C_{M\alpha} U^2/2I = jU^2 \tag{38b}$$

In the present problem the airspeed is not constant, and the role of frequency of oscillation of the relative angle of attack or fundamental frequency is played by

$$\omega_0^2 \equiv bj/f = -(T/mf)\rho c S C_{M\alpha}/2I = -cTC_{M\alpha}/IC_D(\theta_1) \tag{39}$$

which corresponds to Eq. (38b) with the substitution $U^2 \leftrightarrow T/mf$, and can be simplified using Eqs. (15a), (19b), and (26d). The value of ω_0 is weakly dependent on airspeed as for the short period frequency.

Since the oscillation frequency concerns the relative angle of attack, and involves the rotational inertia I, it corresponds to a kind of short-period mode; this is not the short-period mode in the original sense, as the latter arises from linearization of the equations of motion. Therefore, it may be better instead to designate ω_0 as a fundamental frequency of the oscillation of relative angle of attack, which will now be referred to simply as the fundamental frequency. Substituting Eq. (39) in Eq. (36), it follows that the linear terms are similar to Eq. (38a), i.e., represent an oscillation with constant amplitude and frequency ω_0

$$\ddot{\varsigma}[1 + h\varsigma + \varepsilon\varsigma^2] + \omega_0^2\varsigma = \pm ir(\varsigma^{1/2}\ddot{\varsigma}^{-1/2}\dddot{\varsigma} - \varsigma^{-1/2}\dot{\varsigma}\ddot{\varsigma}^{1/2}) \tag{40}$$

although this is modified by the nonlinear terms, involving h, ε, and

$$r \equiv \sqrt{j}/2f = (1/2f)\sqrt{-\rho c S C_{M\alpha}/2I} = [m/C_D(\theta_1)]\sqrt{-C_{M\alpha}c/2\rho S I} \tag{41}$$

where it has been taken into account that static pitch stability requires $C_{M\alpha} < 0$ and Eqs. (19b) and (26d) were used. The sinusoidal oscillation mode [Eq. (38a)] resembles a linear or harmonic oscillator in classical mechanics, but in the present case [Eq. (40)], besides the linear part, there are two sets of nonlinear terms. The first, in square brackets, resembles a nonlinear or anharmonic oscillator, in that it involves powers of the oscillation variable, i.e., the relative angle of attack, but they multiply $\ddot{\varsigma}$ rather than ς; they arise because from h and ε [Eqs. (26a–26c)] and $\bar{f}$, $\bar{g}$, and $\bar{h}$ [Eqs. (15b–15d)] i.e., the nonconstant terms in the lift–drag polar, i.e., the lift-induced drag $\kappa \neq 0$ and nonparabolic $\lambda \neq 0$ lift–drag polar. The second set of nonlinear terms, on the right-hand side, involve products of powers of derivatives of the oscillation variable, and the coefficient involves r [Eq. (41)] and is out of phase by $\pi/2$; since these terms arise from the elimination, for relative angle of attack, between the pitching moment and force equations, it represents, in a loose terminology, a sort of coupling between the short period and phugoid modes. Again, we are not concerned with a true phugoid mode, in the original sense, since the restriction to small flight-path angle limits the phugoid motions possible. It may be more unambiguous to refer to the nonlinear terms as perturbations of the fundamental sinusoidal oscillation of relative angle of attack.

Generation of Growing or Decaying Oscillatory Harmonics

Perturbation methods can be used to show that, to first order, the nonlinear terms do not change the fundamental frequency of oscillation of the relative angle of attack, but generate oscillations at higher frequencies, viz., harmonics or multiples of the fundamental frequency, and introduce, in addition, another mode, whose amplitude may decay or grow with time.

Perturbation Expansion for Relative Angle of Attack and Frequency

The linearization of Eq. (40), viz.,

$$\ddot{\varsigma}_0 + \omega_0^2 \varsigma_0 = 0 \tag{42a}$$

leads to an oscillation of relative angle of attack, with constant amplitude A and fundamental frequency ω_0:

$$\varsigma_0(t) = A \exp(i\omega_0 t) \tag{42b}$$

The sinusoidal function also satisfies the nonlinear terms in curved brackets in Eq. (40), viz.,

$$\varsigma_0^{1/2} \ddot{\varsigma}_0^{-1/2} \dddot{\varsigma}_0 = \varsigma_0^{-1/2} \dot{\varsigma}_0 \ddot{\varsigma}_0^{1/2} \tag{42c}$$

and thus the nonlinear differential equation:

$$\ddot{\varsigma}_0 + \omega_0^2 \varsigma_0 = \pm ir(\varsigma_0^{1/2} \ddot{\varsigma}_0^{-1/2} \dddot{\varsigma}_0 - \varsigma_0^{-1/2} \dot{\varsigma}_0 \ddot{\varsigma}_0^{1/2}) \tag{43}$$

also has the simple solution [Eq. (42b)]; however, this simple solution does not extend to Eq. (40) because of the terms in the square brackets. The latter act like a nonlinear return force

$$\ddot{\varsigma} = -\omega_0^2 \varsigma/(1 + h\varsigma + \varepsilon\varsigma^2) \equiv F(\varsigma) \tag{44}$$

We may take ε as a parameter measuring the importance of nonlinear effects

$$\begin{aligned} \mu \equiv h/\varepsilon\text{:} \quad & [1 + \varepsilon\varsigma(\varsigma + \mu)]\ddot{\varsigma} + \omega^2 \varsigma \\ & = \pm ir(\varsigma^{1/2} \ddot{\varsigma}^{-1/2} \dddot{\varsigma} - \varsigma^{-1/2} \dot{\varsigma} \ddot{\varsigma}^{1/2}) \end{aligned} \tag{45}$$

and μ accounts for a nonparabolic lift–drag polar. Note that the lift–drag polar is parabolic $\mu = 0$, if the term in square brackets in Eq. (45) simplifies to $1 + \varepsilon\varsigma^2$, involving the lift-induced drag. Thus, the nonlinearity parameter ε is defined from the lift-induced drag and can be used whether the lift–drag polar is parabolic or not. The solution of Eq. (45) is sought as a perturbation expansion:

$$\varsigma = \varsigma_0 + \varepsilon\varsigma_1 + \mathcal{O}(\varepsilon^2) \tag{46a}$$

$$\omega = \omega_0 + \varepsilon\omega_1 + \mathcal{O}(\varepsilon^2) \tag{46b}$$

both in the relative angle of attack [Eq. (46a)] and in the fundamental frequency [Eq. (46b)].

Induced Oscillations at Multiples of the Fundamental Frequency

When substituting Eqs. (46a) and (46b), the zeroth-order term, which is independent of ε, coincides with Eq. (43), and hence, vanishes; the first-order term, i.e., the coefficient of ε is

$$\begin{aligned} & \dddot{\varsigma}_1 + [\pm(i/r)\varsigma_0^{-1/2} \ddot{\varsigma}_0^{1/2} - (1/2)(\ddot{\varsigma}_0^{-1} \dddot{\varsigma}_0 + \varsigma_0^{-1} \dot{\varsigma}_0)]\ddot{\varsigma}_1 \\ & \quad - \varsigma_0^{-1} \ddot{\varsigma}_0 \dot{\varsigma}_1 + [\pm(i/r)\omega_0^2 \varsigma_0^{-1/2} \ddot{\varsigma}_0^{1/2} \\ & \quad + (1/2)\varsigma_0^{-1}(\dddot{\varsigma}_0 + \varsigma_0^{-1} \dot{\varsigma}_0 \ddot{\varsigma}_0)]\varsigma_1 = \mp(i/r)[2\omega_0\omega_1 \varsigma_0^{1/2} \ddot{\varsigma}_0^{1/2} \\ & \quad + \omega_0^2 \varsigma_0^{1/2} \ddot{\varsigma}_0^{3/2}(\varsigma_0 + \mu)] \end{aligned} \tag{47}$$

which is a third-order differential equation for the perturbation, with Eq. (42b) appearing in the coefficients

$$\begin{aligned} & \dddot{\varsigma}_1 - \omega_0(\pm 1/r + i)\ddot{\varsigma}_1 + \omega_0^2 \dot{\varsigma}_1 - \omega_0^3(\pm 1/r - i)\varsigma_1 \\ & \quad = \pm(\omega_0^2/r)Ae^{i\omega_0 t}[2\omega_1 - \omega_0 A^{i\omega_0 t}(\mu + Ae^{i\omega_0 t})] \end{aligned} \tag{48}$$

Thus, a forced solution is sought in the form of a superposition of oscillations, at the fundamental frequency plus two harmonics:

$$\varsigma_1(t) = C_1 e^{i\omega_0 t} + C_2 e^{2i\omega_0 t} + C_3 e^{3i\omega_0 t} \tag{49}$$

leading, on the substitution of Eq. (49) into Eq. (48)

$$O.C_1 = 2\omega_1 \omega_0^2 A/r \tag{50a}$$

$$A^2 \mu = -3(1 \mp ir)C_2 \tag{50b}$$

$$A^3 = -8(1 \mp ir)C_3 \tag{50c}$$

From Eq. (50a) it follows that $\omega_1 = 0$, so that there is no frequency shift $\omega = \omega_0$ in Eq. (46b) and no resonant term in Eq. (49), i.e., the first term is included in Eq. (42b) with C_1 being absorbed into A. Note that there would be a frequency shift for the anharmonic oscillator [Eq. (40)], if the nonlinearities in curved brackets were absent. Thus, the perturbation [Eq. (49)] is forced only at the first two harmonics of the fundamental

$$\varsigma_1(t) = -A^2 e^{2i\omega_0 t}[\mu/3 + (A/8)e^{i\omega_0 t}]/(1 \mp ir) \tag{51}$$

the real part is

$$\begin{aligned} \varsigma_1(t) = & -[A^2/(1 + r^2)]\{\mu/3[\cos(2\omega_0 t) \mp r \sin(2\omega_0 t)] \\ & + (A/8)[\cos(3\omega_0 t) \mp r \sin(3\omega_0 t)]\} \end{aligned} \tag{52}$$

and the substitution in Eq. (46a), together with Eq. (42b), specifies the complete oscillation.

Free Relative Angle-of-Attack Oscillation with Damping or Growth in Time

Concerning the free oscillations of Eq. (48), viz.,

$$\dddot{\varsigma}_1 - \omega_0(\pm 1/r + i)\ddot{\varsigma}_1 + \varsigma_0^2 \dot{\varsigma}_1 - \omega_0^3(\pm 1/r + i)\varsigma_1 = 0 \tag{53}$$

a solution is sought in the form of a sinusoidal oscillation

$$\bar{\varsigma}_1(t) = Be^{i\omega t} \tag{54a}$$

where ω satisfies the cubic equation [Eq. (54b)]

$$(\omega^2 - \omega_0^2)[\omega + (-1 \pm i/r)\omega_0] = 0 \tag{54b}$$

Table 1 Calculation of fundamental frequencies and growth rate for McDonnell Douglas F-4 Phantom II[a]

Condition	Approach to land	Cruise		Units
		Subsonic	Supersonic	
M	0.206	0.900	1.800	——
z	0	35	55	$\times 10^3$ ft
U	70.1	267	531	ms^{-1}
C_{MO}	+0.020	+0.025	−0.025	——
$C_{M\alpha}$	−0.098	−0.400	−0.780	rad^{-1}
α_1	+0.204	+0.0625	−0.032	rad
C_{LO}	0.430	0.100	0.010	——
$C_{L\alpha}$	2.80	3.75	2.80	rad^{-1}
α_O	−0.154	−0.0267	−0.00357	rad
θ_1	+0.358	0.0892	−0.0284	rad
C_{DO}	0.0269	0.0205	0.0439	——
$C_{D\alpha}$	0.555	0.300	0.400	rad^{-1}
$C_D(\theta_1)$	0.226	0.0285	0.0325	——
m	15060	17690	17690	kg
ρ	1.293	0.400	0.143	$\mathrm{kg\ m}^{-3}$
f	4.77×10^{-4}	1.58×10^{-5}	6.46×10^{-6}	m^{-1}
α	11.7	2.6	3.3	deg
α	0.204	0.0454	0.0576	rad
$C_D(\alpha)$	0.050	0.0211	0.0452	——
$D = T$	7.815	14.8	44.8	kN
b	0.519	0.837	2.534	ms^{-2}
I	1.913	2.00	2.00	$\times 10^5\ \mathrm{kg\ m}^2$
j	7.95×10^{-5}	9.60×10^{-5}	6.70×10^{-5}	m^{-2}
ω_O	0.294	2.25	5.13	5^{-1}
τ_O	2.14	2.79	1.23	s
ϑ	3.15×10^{-2}	7.27×10^{-3}	8.09×10^{-3}	s^{-1}
τ	22.0	94.9	85.9	s
N	10.3	34.0	69.4	N

[a] $c = 4.88$ m, $S = 49.2\ \mathrm{m}^2$.

As could be expected from $\omega_1 = 0$ and $\omega = \omega_0$ in Eqs. (50a) and (46b), $\pm\omega_0$ are roots of Eq. (54b), and thus, the third root is easily found

$$\omega = \pm\omega_0,\ -\omega_0(1 \pm i/r):$$

$$\varsigma_1(t) = C_+ e^{i\omega_0 t} + C_- e^{-i\omega_0 t} + C_* e^{i\omega_0 t} e^{\pm(\omega_0/r)t} \tag{55}$$

i.e., the motion corresponds to oscillations at the fundamental frequency with constant amplitudes $C_\pm$, plus a term exponentially growing or decaying in time, in proportion to Eqs. (39) and (41)

$$\vartheta \equiv \omega_0/r = 2\sqrt{bf} = 2\sqrt{fT/m} \equiv 0.693/\tau \tag{56}$$

where τ defines the time scale for growth or decay by a factor of 2.

Discussion

The fundamental frequency and the time to halve or double amplitude are calculated next for one statically stable fighter and transport aircraft, and a statically unstable example is also given.

Time to Double or Halve Amplitude

The present method of a solution of the equations of longitudinal motion of a symmetric aeroplane leaves in the elimination [Eqs. (32a) and (32b)] an uncertainty of sign, which persists through the first-order perturbation up to Eq. (55), allowing for modes growing or decaying amplitude, with τ given by Eq. (56) or, using Eq. (26d)

$$0.693/\tau \equiv \vartheta = \omega_0/r = 2\sqrt{bf} = \sqrt{2\rho STC_D(\theta_1)/m} \tag{57}$$

Thus, unstable modes can exist, and an estimate of their growth rate [Eq. (57)] is given by

$$\vartheta = \sqrt{2\rho}\sqrt{T/m}\sqrt{S/m}(C_{D0} + \lambda C_{L\alpha}\theta_1 + kC_{L\alpha}^2\theta_1)^{1/2} \tag{58}$$

In the latter the term in the second square root is larger for lower wing loading, which is a design feature of high-maneuverability air superiority fighters; these also have a high thrust-to-weight ratio, so that the first square root is also larger for these aircraft; the growth rate of instabilities is large for high-drag coefficient (calculated for the angle θ_1 in the second curved brackets), for example, in landing configuration or curved flight at a high turn rate. The fundamental frequency ω_0 and period τ_0 are given by Eqs. (39), (15a), (19b), and (26d)

$$\omega_0^2 = bj/f = -c(T/I)[C_{M\alpha}/C_D(\theta_1)] = (2\pi/\tau_0)^2 \tag{59}$$

and the number of periods to double or halve amplitude by

$$N = \tau/\tau_0 = 0.110r = 0.110[m/C_D(\theta_1)]\sqrt{-cC_{M\alpha}/2\rho SI} \tag{60}$$

where the dimensionless quantity r was used [Eq. (41)]. These quantities are calculated using data from Refs. 9 and 53 in Table 1 for the F-4 and in Table 2 for the B747.

Examples of Fighter and Transport Aircraft

The calculation for a fighter (the McDonnell Douglas F-4 Phantom II) in Table 1 concerns three flight regimes, namely, approach to land and subsonic and supersonic cruise; the corresponding Mach number, true airspeed, and altitude are indicated. From $C_{M\alpha}$ and C_{M0} of the pitching moment coefficient follows Eq. (6b), the angle of attack for zero pitching moment α_1; in a similar way, from the lift coefficient follows Eq. (4b), the α_0, and hence, Eq. (18b), the angle θ_1 of zero pitching moment relative to the angle of zero lift. The drag coefficient at this angle follows from the slope $C_{D\alpha}$ and value at zero angle of attack C_{D0} of the drag coefficient

$$C_D(\theta_1) = C_{D0} + C_{D\alpha}(\theta_1)^2 \tag{61}$$

Using this, the mass m of the aeroplane, the atmospheric ρ, and the wing area S specifies the parameter f [Eq. (26d)], which has the dimensions of inverse length

$$[f] = L^{-1} \tag{62a}$$

Table 2 Calculation of fundamental frequencies and growth rate for Boeing 747[a]

Condition	Approach to land	Cruise Low	Cruise High	Altitude
M	0.198	0.650	0.900	——
z	0	20	40	$\times 10^3$ ft
U	67.4	205	265	ms^{-1}
C_{MO}	0	0	0	——
$C_{M\alpha}$	−1.45	−1.00	−1.60	rad^{-1}
α_1	0	0	0	rad
C_{LO}	0.92	0.21	0.29	——
$C_{L\alpha}$	5.67	4.4	5.5	rad^{-1}
α_O	−0.162	−0.0477	−0.0527	rad
θ_1	+0.162	+0.0477	+0.0527	rad
C_{DO}	0.0269	0.0205	0.0439	——
$C_{D\alpha}$	0.555	0.300	0.400	rad^{-1}
$C_D(\theta_1)$	0.117	0.0348	0.0650	——
m	255830	288778	288778	kg
ρ	1.293	0.689	0.316	$kg\ m^{-3}$
f	1.51×10^{-4}	2.12×10^{-5}	1.82×10^{-5}	m^{-1}
α	8.5	2.5	2.4	0
	0.148	0.044	0.042	rad
$C_D(\alpha)$	0.100	0.0168	0.313	——
$D = T$	150	124	178	kN
b	0.586	0.429	0.616	m^{-2}
I	6.17	7.11	7.11	$\times 10^7$ kg m^2
j	6.46×10^{-5}	2.06×10^{-5}	1.51×10^{-5}	m^{-2}
ω_O	0.500	0.646	0.715	s^{-1}
τ_O	12.5	9.73	8.79	s
ϑ	1.88×10^{-2}	6.03×10^{-3}	6.70×10^{-3}	s^{-1}
τ	36.7	115	103	s
N	2.94	11.8	11.7	——

[a] $c = 8.32$ m, $S = 511$ m^2.

From α and $C_D(\alpha)$ follows the drag [Eq. (3b)], which equals the thrust in steady, straight, and level flight, viz., this will be exactly true for cruise, and approximately so for a stabilized approach with a small flight-path angle. T and m specify b [Eq. (15a)], which has the dimensions of acceleration:

$$[b] = LT^{-2} \tag{62b}$$

I is needed to calculate the parameter j [Eq. (19b)], which has the dimensions of inverse length squared:

$$[j] = L^{-2} \tag{62c}$$

as follows from Eq. (23a). The three parameters f, b, and j specify the fundamental frequency ω_0 and period τ_0 [Eq. (59)], and the growth rate ϑ and timescale τ [Eqs. (57) and (58)], and hence, the latter in terms of oscillation periods N [Eq. (60)]. The calculations for a large transport (Boeing 747) in Table 2 concern again three flight regimes, viz., approach to land, and cruise at low or high altitude. The calculations are similar to Table 1, with the simplification that the pitching moment is zero at zero angle of attack. In all cases the fundamental period is short, viz., 1–3 s for a fighter and longer, 8–13 s, for a transport; the growth time is much longer, viz., 20–120 s, so that the number of fundamental periods to double or halve amplitude is large, viz., $N \sim 3$ to 12 for the transport and $N \sim 10$ to 70 for the fighter, meaning that it is a slow-growing instability, with plenty of time for compensation, even by manual control.

Example of a Statically Unstable Aircraft

Consideration has been given so far to statically stable aircraft $C_{M\alpha} < 0$, for which in Eq. (19b) the parameter j is positive, $j > 0$, and thus, the fundamental frequency [Eq. (59)] real. In the case of a statically unstable aircraft $C_{M\alpha} > 0$, then $j < 0$

$$C_{M\alpha} > 0: \quad j = -|j| \tag{63a}$$

and the fundamental frequency is imaginary

$$\omega_0 = \sqrt{bj/f} = \pm i\sqrt{b|j|/f} = \pm i|\omega_0| \tag{63b}$$

so that now the roles of ω_0 and ϑ are interchanged

$$\exp[\omega_0(i \pm 1/r)t] = \exp[\mp|\omega_0|t + i(|\omega_0|/r)t] \tag{64}$$

i.e., $\bar{\omega}_0 = |\omega_0|/r$ is the oscillation frequency and $\bar{\vartheta} = |\omega_0|$ the growth or decay rate; taking as an example data[54] for the F-16C, which has a takeoff mass of $m = 12{,}040$ kg, and afterburning thrust $T = 106.3$ kN, corresponding to a thrust-to-weight ratio $T/W = T/mg = 0.90$. The wing area is $S = 27.88$ m^2, and the span $\bar{b} = 9.45$ m, corresponds to a mean chord $c = \bar{b}^2/S = 3.20$ m. Taking $C_D(\theta_1) = 0.25$ as for the F-4, the oscillation frequency is $\bar{\omega}_0 = |\vartheta| = 0.460$ s^{-1}, corresponding to a period $\bar{\tau}_0 = 2\pi/\bar{\omega}_0 = 13.7$ s. Using $\mu \equiv IC_0(\theta_1)/|C_{M\alpha}| = 4.4 \times 10^5$ kg m^2 as for the F-4, the growth rate $\bar{\vartheta} = |\omega_0| = 0.88$ s^{-1} and time to double amplitude $\bar{\tau} = 0.693/\bar{\upsilon} = 0.79$ s show that the instability is rapid.

The original analysis of the phugoid[13] uses the force balance equations[12] [Eqs. (1a) and (1b)] with constant thrust equal to drag and constant angle of attack:

$$m\dot{U} = -W \sin \varsigma \tag{65a}$$

$$mU\dot{\gamma} = L - W \cos \gamma \tag{65b}$$

A complementary short-period analysis would use the pitching moment equation [Eqs. (2), (3c), and (6a)]

$$\ddot{\gamma} << \ddot{\alpha}: \quad \ddot{\alpha} - (C_{M\alpha}\rho cS/2I)U^2(\alpha - \alpha_1) = 0 \tag{66a}$$

and also the condition of thrust equal to drag [Eq. (3b)]

$$T = D = \tfrac{1}{2}\rho SU^2 C_D(\alpha) \tag{66b}$$

The equation of angle-of-attack oscillations [Eq. (66a)] is nonlinear because U depends on angle of attack through Eq. (66b); it is linearized by evaluating airspeed at the mean state, i.e., angle of attack of zero pitching moment:

$$U_1 = \sqrt{2T/\rho SC_{D\alpha}(\alpha_1 - \alpha_0)} \qquad (67)$$

Substituting Eq. (67) in Eq. (66a) yields

$$\varsigma = \alpha - \alpha_1: \quad \ddot{\varsigma} - [C_{M\alpha}cT/IC_D(\theta_1)]\varsigma = 0 \qquad (68)$$

which coincides with Eqs. (42a) and (39), and specifies 1) a sinusoidal oscillation of frequency ω_0 for statically stable aircraft $C_{M\alpha} < 0$ and 2) an amplitude growth or decay with time-to-double or halve of $0.693/|\omega_0|$, for a statically unstable aircraft $C_{M\alpha} > 0$. The determination of the other parameter r [Eq. (41)] or ϑ [Eqs. (57) and (58)] requires a nonlinear analysis, beyond the simple deduction [Eqs. (66–68)], which complements the phugoid problem.

Conclusions

The last two decades have seen major progress in flight control technology, as witnessed by improvement in handling qualities of the current fighters (F-14 Tomcat, F-15 Eagle, F-16 Falcon, and the F-18 Hornet), compared with the century series (F-100 Super Sabre, F-101 Voodoo, F-102 Delta Dagger, F-104 Starfighter, F-105 Thunderchief, F-106 Delta Dart, F-111 Aardward and also the F-8 Crusader and F-4 Phantom II). Statically unstable designs give improved maneuverability, and allow a smaller design for the same mission, and active control technology can provide gust alleviation and load limitation as well as protection from departure at high angle of attack and/or sideslip, yielding an expanded carefree maneuver envelope. Modern control technology also brought some partially unsolved problems, like the PIO, which has caused accidents of both manned (F-22, Gripen) and unmanned (Darkstar) aircraft. Maybe the progress in flight control technology should be matched by advances in flight dynamics, into the unsteady and nonlinear regimes, which have received less attention in the literature, but may hold the key to a better understanding, not only of spins, but perhaps also of PIOs.

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